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SINGULAR DIRECTIONS IN THE CONFIGURATION SPACE OF LINEAR VIBRATING SYSTEMS†

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Besides the familiar concept of principal directions of normal modes in the theory of oscillations of linear systems with constant coefficients, new notions are introduced: directions conjugate to the principal directions, and principal directions of forced vibrations. The basic properties of vibrations in these directions are established.

1. WE CONSIDER a real system

$$Ax^{\cdot\cdot} + Bx = p \cos \omega t \tag{1.1}$$

where A and B are symmetric positive definite $n \times n$ matrices, x is an n-vector written as a column, p an n-vector, also written as a column, whose modulus represents the amplitude of a periodic driving force and its direction is the direction in which this force is applied in the configuration space.

The right-hand side of (1.1) is a special case of a periodic force driving a system with n degrees of freedom, in which the forces acting on each degree of freedom are synchronous and in phase. Later we shall also consider a more-general situation.

For the sake of continuity, we will first outline a few prerequisites [1].

If $p = 0$, a particular solution of system (1.1) is sought in the form $x = q \cos \nu t$, which leads to the following algebraic system:

$$(B - \nu^2 A)\, q = 0 \tag{1.2}$$

A necessary and sufficient condition for this system to have a non-trivial solution q is that

$$\det (B - \nu^2 A) = 0 \tag{1.3}$$

Equation (1.3) has n positive roots $\nu_1, \ldots, \nu_n$, corresponding to which are n solutions of system (1.2): $q_1, \ldots, q_n$. The directions defined by these vectors in the configuration space are known as the principal directions. The vectors q_k have the property of A-orthogonality:

$$(q_k, Aq_l) = \begin{cases} 0, & k \neq l \\ 1, & k = l \end{cases}$$

The principal directions are singular in the sense that only in these directions can one obtain longitudinal modes of vibration, which moreover take place at only one frequency.

2. Let us associate with each principal direction q_k a conjugate singular direction Bq_k. [The vector Aq_k defines the same direction, since by (1.2), $Bq_k = \nu_k^2 Aq_k$.] The properties of conjugate directions are established by the following theorem.

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Theorem 1. If the force $p\cos\omega t$ in system (1.1) acts in a conjugate direction, the system will vibrate with frequency ω in a principal direction; and as ω varies from zero to infinity the system will behave as if it were one-dimensional, i.e. the amplitude-frequency characteristic (AFC) will have a single discontinuity of the second kind at the point $\omega = \nu_k$ when $p = Bq_k$.

Proof. Let us seek a solution with the frequency of the driving force, in the form $q = aq_k\cos\omega t$, where a is a scalar defining the amplitude of the solution. Substituting this solution into system (1.1), we find

$$a(B - \omega^2 A)\, q_k = Bq_k$$

Substituting $Aq_k = \nu_k^{-2}Bq_k$ into this equation, we get

$$[a(1 - \omega^2/\nu_k^2) - 1]\, Bq_k = 0$$

whence $a = \nu_k^2/(\nu_k^2 - \omega^2)$. The theorem is proved.

The "singular" property of the directions conjugate to the principal directions is that whenever the driving force deviates by an arbitrarily small amount from a conjugate direction, n points of discontinuity may appear in the AFC.

For example, if the driving force is $p\cos\omega t = Aq_k\cos\nu_l t$, where $k \neq l$—that is, the force acts in the direction of Aq_k but with one of the natural frequencies of the system other than ν_k—then the periodic solution $q\cos\nu_l t$ will have a finite amplitude $a = \nu_k^2/(\nu_k^2 - \nu_l^2)$, but this amplitude will become infinite if the direction of the force is perturbed, albeit by an arbitrarily small amount: $p = Aq_k + \delta p$.

The theorem can be generalized as follows.

Theorem 2. If the force $p\cos\omega t$ in system (1.1) lies in the linear subspace spanned by m vectors of conjugate directions Bq_{k_s} $(s = 1, m \leqslant n)$, i.e. $p = B(b_1 q_{k_1} + \ldots + b_m q_{k_m})$, then the motion of the system at frequency ω will belong to the linear space spanned by the principal direction vectors q_{k_s}, i.e. $q = a_1 q_{k_1} + \ldots + a_m q_{k_m}$, and as ω varies from zero to infinity the system will behave like a system with m degrees of freedom (the AFC will show m resonances at the points $\nu_{k_1}, \ldots, \nu_{k_m}$).

The proof is a natural generalization of that of Theorem 1.

3. *Definition.* Let $x = q\cos\omega t$ be a particular solution of system (1.1), defining forced vibrations driven by a force $p\cos\omega t$. A principal direction of forced modes (PDFM) is defined as a direction q which coincides with the direction of the force p, i.e.

$$q = \lambda p \tag{3.1}$$

where λ is a scalar defining the amplitude of the vibrations.

Since the vector q is always related to p by the formula $(B - \omega^2 A)q = p$, a PDFM must satisfy the condition

$$(B - \omega^2 A)\, q = \lambda^{-1} q \tag{3.2}$$

i.e. a vector that defines a principal direction is an eigenvector of the matrix $B - \omega^2 A$, belonging to an eigenvalue which is the reciprocal of the amplitude $\mu = \lambda^{-1}$. This eigenvalue is a root of the equation

$$\det(B - \omega^2 A - \mu E) = 0 \tag{3.3}$$

Since the matrix $B - \omega^2 A$ is symmetric, this equation has exactly n real solutions. Since A and B are positive definite, there exists $\omega_{\min}$ such that $B - \omega^2 A$ is also positive definite for all ω in the interval $0 < \omega < \omega_{\min}$. At the same time there also exists $\omega_{\max}$ such that $B - \omega^2 A$ is negative definite for all ω in the interval $\omega_{\max} < \omega < \infty$.

The first case is referred to as pre-resonant; when it occurs, all the values of μ are positive. The second is the post-resonant case, when all the values of μ are negative. Since the passage from negative to positive μ values involves going through zero, $\omega_{\min}$ and $\omega_{\max}$ are the least and greatest roots, respectively, of the equation

$$\det (B - \omega^2 A) = 0$$

To each solution μ of Eq. (3.3) there corresponds an eigenvector q—a solution of Eq. (3.2). Thus, there exist exactly n PDFMs $q_1, \ldots, q_n$ of system (1.1). These vectors form an orthogonal basis: $(q_i, q_j) = 0$ if $i \neq j$. The reader should note that the PDFMs coincide neither with the principal directions of the normal modes nor with the conjugate directions. If the principal directions of the normal modes have the A-orthogonality property: $(q_i, Aq_j) = 0$ for $i \neq j$, and the conjugate directions have the A^{-1}-orthogonality property: $(q_i, A^{-1}q_j) = 0$ for $i \neq j$, then the PDFMs are orthogonal in the sense of the ordinary Euclidean metric.

4. Equation (3.3) defines an implicit function $\mu(\omega^2)$ with n branches. The properties of this function are clarified in the following theorem.

Theorem 3. Each branch of the implicit function (3.3) is a monotone decreasing function that has exactly one zero in the interval $\omega^2 \in [0, \infty)$.

Proof. Taking the scalar product of Eq. (3.2) and q, we obtain a scalar equation

$$(q, Bq) - \omega^2 (q, Aq) - \mu (q, q) = 0 \tag{4.1}$$

In this equation, vectors q that are solutions of Eq. (3.2) are functions of ω^2. Hence, differentiating (4.1) with respect to ω^2, we obtain

$$2\left(\frac{dq}{d\omega^2}, Bq\right) - (q, Aq) - 2\omega^2\left(\frac{dq}{d\omega^2}, Aq\right) - \frac{d\mu}{d\omega^2}(q, q) - 2\mu\left(\frac{dq}{d\omega^2}, q\right) = 0$$

Since the terms in this equation that contain $dq/d\omega^2$ vanish, in view of (3.2), the remaining terms give

$$d\mu/d\omega^2 = -(q, Aq)/(q, q) \tag{4.2}$$

Since A is positive definite, this implies $d\mu/d\omega^2 < 0$.

Note that in formula (4.2) each eigenvector determines the derivative of the corresponding eigenvalue μ. If Eq. (3.3) has multiple roots for some ω^2, then, since the matrix $B - \omega^2 A$ is symmetric, the eigenvalues are still differentiable functions of ω^2 and formula (4.2) remains true, but it requires a special choice of eigenvectors from the eigenspace corresponding to the multiple root. At any rate, we have proved that $\mu(\omega^2)$ is monotone decreasing. We also note that $\mu(0) > 0$, since $\mu(0)$ are eigenvalues of B. On the other hand, $\lim_{\omega^2 \to \infty} \mu(\omega^2)/\omega^2 < 0$, since this limit is an eigenvalue of the matrix $-A$. A monotone decreasing continuous function that has different signs at either end of an interval has just one zero in the interior of the interval.

This completes the proof of the theorem.

Figure 1 shows an arbitrary branch of the function $\mu(\omega^2)$. The quantity $|\lambda(\omega^2)| = |\mu(\omega^2)|^{-1}$ represents the AFC of principal forced modes of vibration of system (1.1).

It follows from the aforesaid that any system (1.1) has exactly n AFCs of principal vibrations, each of which has a unique point of discontinuity of the second kind. This means that if a system with n degrees of freedom is driven along a principal direction in the sense defined above, and the applied frequency is varied from 0 to ∞, then one resonance will be observed. As in the case of a force applied along a conjugate direction, the system will behave as though it were one-dimensional. To each principal direction there corresponds a specific resonance frequency, which is simply one of the natural frequencies of the system.

5. If there are dissipative forces in the system, we rewrite system (1.1) as follows:

$$A\ddot{x} + D\dot{x} + Bx = pe^{i\omega t} \tag{5.1}$$

where D is a positive definite matrix and p a column vector with complex coordinates.

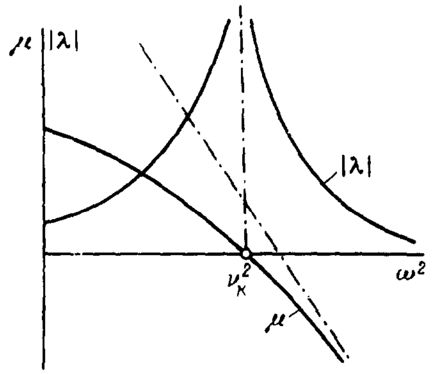

FIG. 1.

Unlike the previous case, periodic forces acting on different coordinates, even if synchronous, will not necessarily be in phase.

Our definition of PDFM may be generalized naturally as follows.

Definition. Let $x = qe^{i\omega t}$ be a periodic solution of system (5.1), where q is a column vector with complex coordinates. A PDFM in system (5.1) is a direction of q at which it coincides with that of p: $q = \lambda p$, where λ is a complex number.

The word "direction" is being used here in a somewhat non-standard sense; what we mean is the following. The real part of system (5.1) has the form (summation over s):

$$a_{ks}x_s^{\cdot\cdot} + d_{ks}x_s^{\cdot} + b_{ks}x_s = r_k \cos(\omega t + \varphi_k) \tag{5.2}$$

In the configuration space of system (5.2), the right-hand side describes an ellipse in a certain two-dimensional subspace. By the direction of the force in this case we mean the orientation in the configuration space of the semi-axes of the ellipse, the ratio between the semi-axes and the sense in which the ellipse is described.

To say that the direction of the force coincides with that of a forced mode means that the point x in the configuration space of system (5.2) describes an ellipse which is the same, apart from similitude, as the ellipse of the force. In motion around the ellipse itself, the displacement vector lags behind the force vector, and this constant phase shift is determined by the ratio between the imaginary and real parts of the complex amplitude coefficient λ (Fig. 2).

A vector p defining a PDFM is found from the system

$$(-\omega^2 A + i\omega D + B - \mu E)\, p = 0 \tag{5.3}$$

where the complex eigenvalue μ, as before, is the reciprocal of the amplitude factor λ.

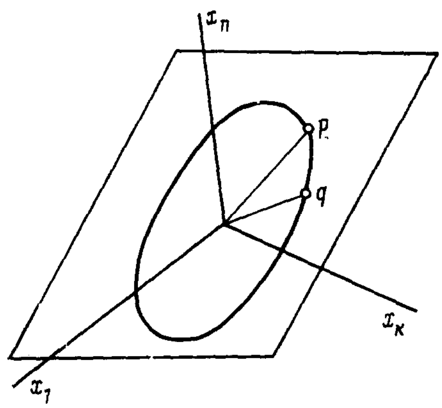

FIG. 2.

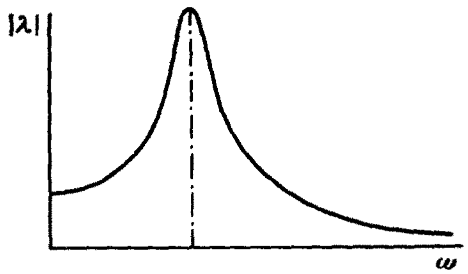

FIG. 3.

Equating the determinant of system (5.3) to zero, we get an implicit function $\mu(\omega)$ with n branches. Multiplying (5.3) on the left by its Hermitian conjugate, as in (4.2), we obtain

$$d\mu/d\omega = [-2\omega\,(p^*, Ap) + i\,(p^*, Dp)]/(p^*, p)$$

whence it follows that the derivative of the real part of the root is negative:

$$d\,(\mathrm{Re}\,\mu)/d\omega = -2\omega\,(p^*, Ap)/(p^*, p) < 0$$

At $\omega = 0$ the real part of the root is positive, but as $\omega \to \infty$ it becomes negative; it follows that it will vanish just once at some ω. Hence the AFC of principal forced modes in a system with fairly small dissipation will have the form shown in Fig. 3.

It should be noted that in a system with dissipation the PDFMs are no longer singular: if the direction of the force deviates slightly from a principal direction, the AFC shown in Fig. 3 will also vary only slightly.

6. To include gyroscopic forces in our treatment, it will be convenient, as in the case of dissipative forces, to write the periodic force in complex form:

$$Ax^{\cdot\cdot} + \Gamma x^{\cdot} + Bx = pe^{i\omega t} \tag{6.1}$$

where Γ is the skew-symmetric matrix of gyroscopic forces. The meaning of the principal directions here is the same as in the dissipative case. Writing the unknown solution of system (6.1) as $x = qe^{i\omega t}$ and setting $q = \lambda p$, we obtain

$$(-A\omega^2 + i\omega\Gamma + B - \mu E)\,p = 0 \tag{6.2}$$

Multiplying this equation on the left by its Hermitian conjugate and solving the resulting equation for μ, we get

$$\mu = [-\omega^2\,(p^*, Ap) + i\omega\,(p^*, \Gamma p) + (p^*, Bp)]/(p^*, p)$$

Since the products (p^*, Ap), $i(p^*, \Gamma p)$, (p^*, Bp), (p^*, p) are all real numbers, it follows that for any skew-symmetric matrix Γ the root μ is a real number. This means that in motion around the ellipse (Fig. 2) the phase shift between the force and the displacement is either zero or π. Each branch of $\mu(\omega)$ will vanish exactly once, so that the AFC of PDFMs in a gyroscopic system will have one discontinuity ofthe second kind.

The case of gyroscopic forces differs from those considered previously in that $\mu(\omega)$ is no longer a monotone decreasing function.

Let us consider, as an example, a gyroscopic system of fourth order. We choose the matrices A, B, Γ thus:

$$A = B = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, \quad \Gamma = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}$$

System (6.2), from which we can find the principal directions, becomes

$$\begin{Vmatrix} 1 - \omega^2 - \mu & i\omega \\ -i\omega & 1 - \omega^2 - \mu \end{Vmatrix} p = 0 \tag{6.3}$$

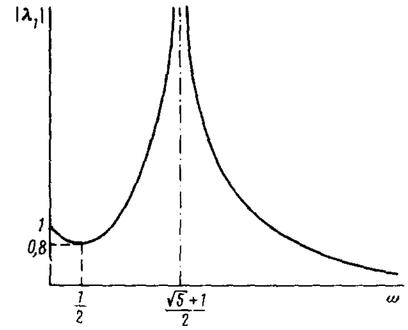

FIG. 4.

Equating the determinant of this system to zero, we find the roots

$$\mu_1 = 1 + \omega - \omega^2, \quad \mu_2 = 1 - \omega - \omega^2$$

The AFC of the first principal direction $|\lambda_1| = |\mu_1|^{-1}$ is shown in Fig. 4. It is obvious that since μ_1 is no longer monotone, a minimum appears in the interval $0<\omega<(\sqrt{5}+1)/2$.

The AFC of the second principal direction is qualitatively the same as that of an ordinary one-dimensional system without dissipation.

The solutions of system (6.3), which determine the principal directions, are

$$p_1 = \left\| \begin{matrix} i \\ 1 \end{matrix} \right\|, \quad p_2 = \left\| \begin{matrix} -i \\ 1 \end{matrix} \right\|$$

Hence it follows that the gyroscopic system ($p = p_1$)

$$x^{\cdot\cdot} + y^{\cdot} + x = -\sin \omega t, \quad y^{\cdot\cdot} - x^{\cdot} + y = \cos \omega t \tag{6.4}$$

has a periodic solution

$$x = -\sin \omega t/(1 + \omega - \omega^2), \quad y = \cos \omega t/(1 + \omega - \omega^2)$$

which is proportional to the driving force and has only one singularity as ω is varied [at the point $\omega = (\sqrt{5}+1)/2$].

If $p = p_2$, however, we get a system analogous to (6.4) but with ω replaced by $-\omega$, and the periodic solution has a singularity at $(\sqrt{5}-1)/2$.

7. The effect of intrinsically non-conservative forces. Suppose that system (1.1) involves, besides positional conservative forces, certain non-conservative forces with a skew-symmetric matrix N:

$$Ax^{\cdot\cdot} + (B + N)\, x = p \cos \omega t \tag{7.1}$$

If $\|N\| < \min_{k,l} |\omega_k^2 - \omega_l^2|\ (n\sqrt{2})$, where ω_k^2 are the roots of the equation $\det(B - \omega^2 A) = 0$, then the homogeneous part of the system has real natural frequencies, i.e. it conserves the oscillatory nature of the solutions. This is the case of interest here.

Seeking a solution of system (7.1) in the form $x = q \cos \omega t$, we obtain

$$(B + N - \omega^2 A - \lambda E)\, q = 0 \tag{7.2}$$

Let us substitute $q = Lr$ into this equation, where L is an orthogonal matrix so chosen that $L^T AL = E$. Then Eq. (7.2) becomes

$$(B^\circ - (\omega^2 + \lambda)\, E)\, r = 0 \quad (B^\circ = L^T (B + N)\, L) \tag{7.3}$$

We now multiply both sides of (7.3) on the left by a symmetric matrix S:

$$[SB^\circ - (\omega^2 + \lambda)\, S]\, r = 0 \tag{7.4}$$

choosing this matrix so that SB° is symmetric

$$SB^\circ = B^{\circ T} S \tag{7.5}$$

The matrix equation (7.5) has an infinite number of solutions S. By virtue of the restrictions imposed on $\|N\|$, we can choose positive definite matrices among these solutions. Since SB° and S are symmetric positive definite matrices, system (7.4) has n real eigenvalues $\lambda_1, \ldots, \lambda_n$ and n real vectors $r_1, \ldots, r_k$. The PDFMs in system (7.3) are obtained as $q_k = Lr_k$. They are orthogonal in the sense of the metric defined by the matrix LSL^T.

The properties of the AFC of the PDFMs for this case are the same as in the case of purely conservative forces.

8. We will now consider the general case

$$Ax^{\cdot\cdot} + (D + \Gamma)\,x^{\cdot} + (B + N)\,x = pe^{i\omega t} \tag{8.1}$$

Here A, B, and D are symmetric matrices and Γ and N are skew-symmetric. As everywhere previously, there are n complex PDFMs $q_1, \ldots, q_n$, which are solutions of the algebraic system

$$(-\omega^2 A + i\omega\,(D + \Gamma) + B + N)\,q = \lambda^{-1} q$$

Any vector p in (8.1) may be expressed as a linear combination of the principal directions: $p = b_1 q_1 + \ldots + b_n q_n$.

Accordingly, a periodic solution may always be expressed as a linear combination of vibrations in the principal directions:

$$x = [\lambda_1(\omega)\, b_1 q_1 + \ldots + \lambda_n(\omega)\, b_n q_n]\, e^{i\omega t} \tag{8.2}$$

If $D = 0$ and the above conditions are satisfied (A and B are positive definite and the norm $\|N\|$ is bounded), then the AFC of each principal forced mode $|\lambda_k(\omega)|$ involves exactly one point of discontinuity at $\omega = \nu_k$, where ν_k are the natural frequencies of the homogeneous part of the system: $\det(-\nu^2 A + i\nu\Gamma + B + N) = 0$. But if these conditions are not satisfied, or if $D \neq 0$, there may be less than n characteristics with singularities (or none at all).

We will conclude with the following remark. The general form of a periodic solution of system (8.1) is

$$x = [-\omega^2 A + i\omega\,(D + \Gamma) + B + N]^{-1}\, pe^{i\omega t}$$

The inverse matrix occurring on the right exists provided that $\omega \neq \nu_k$ ($k = 1, \ldots, n$). The coefficient of $e^{i\omega t}$ is a rational function of ω and, by an elementary theorem, may be expressed as the sum of simple fractions. The analogous coefficient in (8.2), is qualitatively speaking, a different type of expansion, underlying which is the above definition of principal directions. The coefficients $\lambda_k(\omega)$ in (8.2) are not simple fractions: they are meromorphic functions with at most one simple singularity.

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Translated by D.L.